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MT129 - Calculus and Probability

Course Contents

- Functions
- Derivatives
- Applications of Derivative
- Techniques of Differentiation
- Logarithmic Functions and Applications
- The Definite Integrals
- The Trigonometric Functions
- Probability and Calculus

Tutorial 1

Functions

Outline

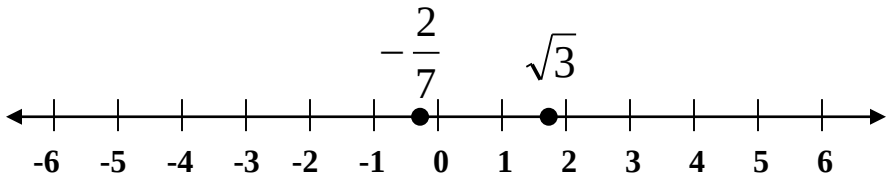
- Functions and Their Graphs
- Some Important Functions
- The Algebra of Functions
- Zeros of Functions
- The Quadratic Formula and Factoring
- Exponents and Power
- Geometric Problems

Real (Rational & Irrational) Numbers

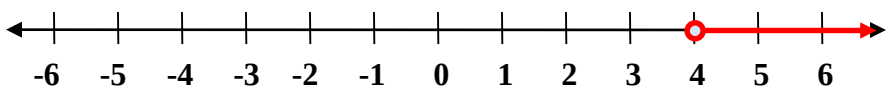
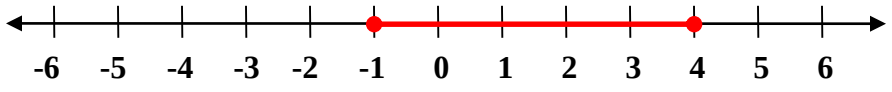
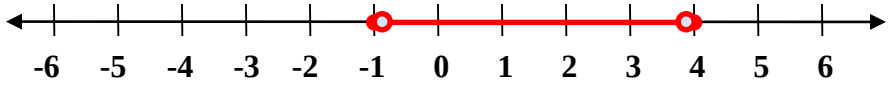
Definition	Example
Rational Number: A number that may be written as a finite or infinite repeating decimal, in other words, a number that can be written in the form m/n such that m, n are integers	• $\frac{2}{7} = 0.285714\,285714\ldots = 0.\overline{285714}$ • $-\frac{1}{2} = -0.5$
Irrational Number: A number that has an infinite decimal representation whose digits form no *repeating* pattern	$\sqrt{3} = 1.73205\ldots$

The Number Line

A geometric representation of the real numbers is shown below

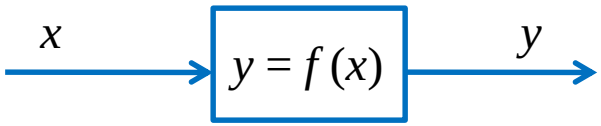


Infinite, Open & Closed Intervals

Definition	Example
Infinite Interval: The set of numbers that lie between a given endpoint and the infinity	 $(4, \infty)$ $x > 4$
Closed Interval: The set of numbers that lie between two given endpoints, including the endpoints themselves	 $[-1, 4]$ $-1 \leq x \leq 4$
Open Interval: The set of numbers that lie between two given endpoints, *not* including the endpoints themselves	 $(-1, 4)$ $-1 < x < 4$

Functions

- A **function** f is a rule that assigns to each value of a real variable x exactly one value of another real variable y.
- The variable x is called the **independent variable** and the variable y is called the **dependent variable**.
- We usually write $y = f(x)$ to express the fact that y is a function of x. Here $f(x)$ is the **name** of the function.



EXAMPLES:

$$f(x) = x^2 + 4$$

$$g(z) = \frac{1}{z-3}$$

$$s(t) = \sqrt{t^2 - 1}$$

$$k(\theta) = e^{(-\frac{2}{\pi}\theta)} + \sin(\theta)$$

Functions

EXAMPLE

If $f(x) = x^2 + 4x + 3$, find $f(-2)$ and $f(a - 2)$.

SOLUTION

$f(x) = x^2 + 4x + 3$	This is the given function.
$f(-2) = (-2)^2 + 4(-2) + 3 = -1$	Replace each x with -2.
$f(a-2) = (a-2)^2 + 4(a-2) + 3$	Replace each x with $a - 2$.
$f(a-2) = (a^2 - 4a + 4) + 4(a-2) + 3$	Evaluate $(a - 2)^2 = a^2 - 4a + 4$.
$f(a-2) = a^2 - 4a + 4 + 4a - 8 + 3$	Remove parentheses and distribute.
$f(a-2) = a^2 - 1$	Combine like terms.

Functions

EXAMPLE

Let $R(x) = \dfrac{100x}{b+x}$.

(a) Suppose that $b = 20$. Find R when $x = 60$.

(b) Determine the value of b if $R(50) = 60$.

SOLUTION

(a) $R(60) = \dfrac{100 \cdot 60}{20+60} = 75$

Replace b with 20 and x with 60.

Therefore, when $b = 20$ and $x = 60$, $R(x) = 75$.

CONTINUED

(b) $R(x) = \dfrac{100x}{b+x}$

This is the given function.

$R(50) = \dfrac{100 \cdot 50}{b+50}$

Replace x with 50.

$60 = \dfrac{100 \cdot 50}{b+50}$

Replace $R(50)$ with 60.

$(b+50) \cdot 60 = \dfrac{5000}{b+50} \cdot (b+50)$

Multiply both sides by $b + 50$.

$60b + 3000 = 5000$

Distribute on the left side.

$60b = 2000$

Subtract 3000 from both sides.

$b = 33.3$

Divide both sides by 60.

Therefore, $b = 33.3$ when $R(50) = 60$.

Domain of a Function

Definition	Example
Domain of a Function: The set of acceptable values for the variable x.	The domain of the function $f(x) = x^4 - 3x^2 + 6$ is $\boldsymbol{R}$ $-\infty < x < \infty$ $(-\infty, \infty)$

Domain of a Function

Definition	Example
Domain of a Function: The set of acceptable values for the variable x.	The domain of the function $f(x)=\sqrt{x+2}$ is $x+2\geq 0$ $x\geq -2$ $[-2,\infty)$

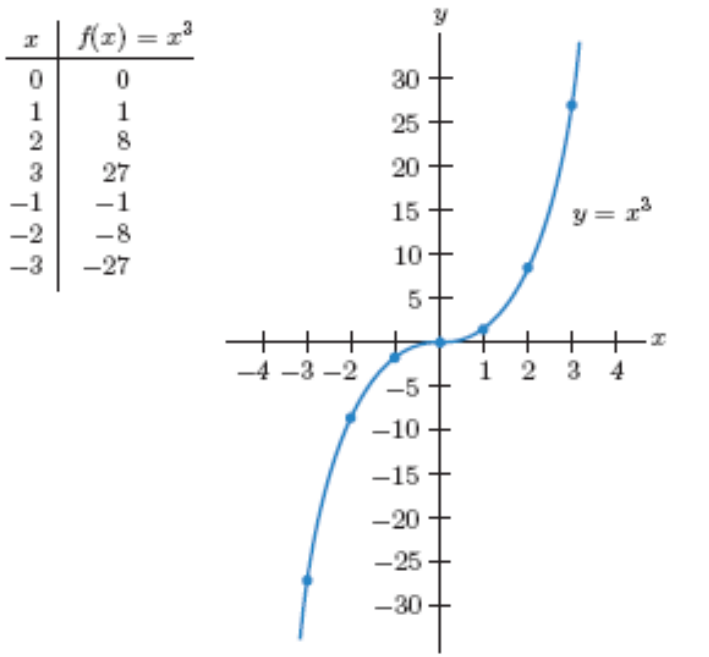
Domain of a Function

Definition	Example
Domain of a Function: The set of acceptable values for the variable x.	The domain of the function $f(x) = \frac{2x}{x^2 - 4}$ is $x^2 - 4 \neq 0$, $x \neq \pm 2$, $x \in R \setminus \{\pm 2\}$

Domain of a Function

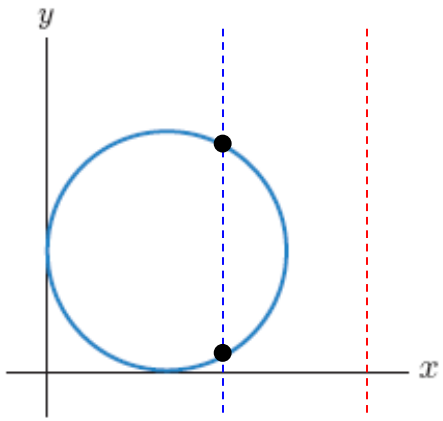
Definition	Example
Domain of a Function: The set of acceptable values for the variable x.	The domain of the function $$f(x)=\frac{x^2}{\sqrt{3-x}}$$ is $$3-x>0$$ $$x<3$$ $$(-\infty, 3)$$

Graphs of Functions

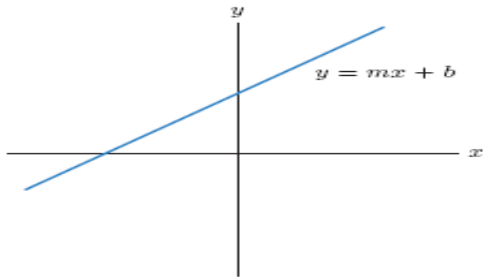
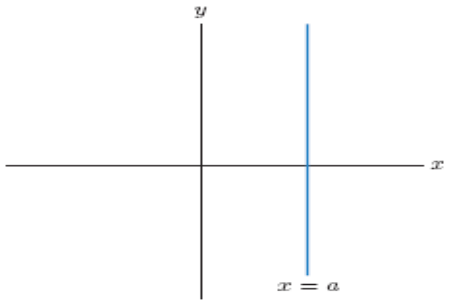
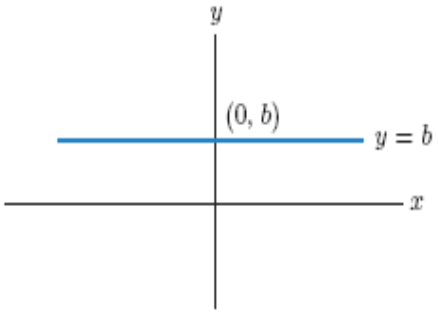
Definition	Example
Graph of a Function: The set of all points $(x, f(x))$ where x is the domain of $f(x)$. Generally, this forms a curve in the xy-plane.	

x	$f(x) = x^3$
0	0
1	1
2	8
3	27
−1	−1
−2	−8
−3	−27

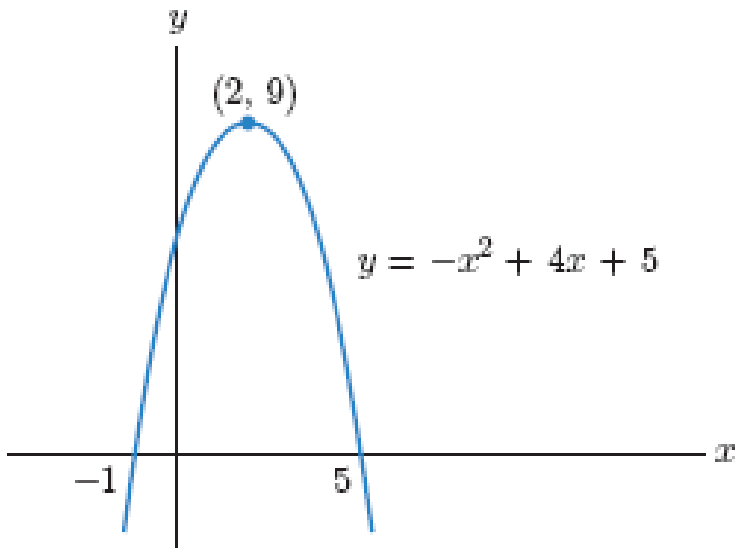
The Vertical Line Test

Definition	Example
Vertical Line Test: A curve in the *xy*-plane is the graph of a function if and only if *each* vertical line cuts or touches the curve at *no more than one point*.	 Although the red line intersects the graph no more than once (not at all in this case), there does exist a line (the yellow line) that intersects the graph more than once. Therefore, this is *not* the graph of a function.

Linear Equations

Equation	Example
$y = mx + b$ (*This is a linear function*)	
$x = a$ (*This is <u>not</u> the graph of a function*)	
$y = b$ (*This is a constant function*)	

Quadratic Functions

Definition	Example
Quadratic Function: A function of the form $f(x) = ax^2 + bx + c$ where a, b, and c are constants and $a \neq 0$.	

Polynomial Functions

Definition	Example
Polynomial Function: A function of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ where n is a nonnegative integer and $a_0, a_1, \ldots, a_n$ are given numbers.	$f(x) = -17x^3 + x^2 + 5$

Rational Functions

Definition	Example
Rational Function: A function expressed as the quotient of two polynomials.	$g(x) = \dfrac{3x - x^4}{5x^2 + x - 1}$

Power Functions

Definition	Example
Power Function: A function of the form $f(x) = x^r$.	$f(x) = x^{5.2}$

Absolute Value Function

Definition	Example
Absolute Value Function: The function defined for all numbers x by $$f(x) = \lvert x \rvert,$$ such that $\lvert x \rvert$ is understood to be x if x is positive and $-x$ if x is negative	$$f(x) = \lvert x \rvert$$ $$f(-1/2) = \lvert -1/2 \rvert = 1/2$$

Adding Functions

EXAMPLE

Given $f(x)=\dfrac{2}{x-3}$ and $g(x)=\dfrac{1}{x+2}$, express $f(x) + g(x)$ as a rational function.

SOLUTION

$f(x) + g(x) =$

$$=\frac{2}{x-3}+\frac{1}{x+2}$$

Replace $f(x)$ and $g(x)$ with the given functions.

$$=\frac{x+2}{x+2}\cdot\frac{2}{x-3}+\frac{1}{x+2}\cdot\frac{x-3}{x-3}$$

Multiply to get common denominators.

$$=\frac{2x+4}{(x+2)(x-3)}+\frac{x-3}{(x+2)(x-3)}$$

Evaluate.

$$=\frac{2x+4+x-3}{(x+2)(x-3)} \qquad =\frac{3x+1}{(x+2)(x-3)}$$

Add and simplify the numerator.

$$=\frac{3x+1}{x^2-x-6}$$

Evaluate the denominator.

Subtracting Functions

EXAMPLE

Given $f(x)=\dfrac{2}{x-3}$ and $g(x)=\dfrac{1}{x+2}$, express $f(x)-g(x)$ as a rational function.

SOLUTION

$f(x)-g(x)=$

$$=\frac{2}{x-3}-\frac{1}{x+2}$$

Replace $f(x)$ and $g(x)$ with the given functions.

$$=\frac{x+2}{x+2}\cdot\frac{2}{x-3}-\frac{1}{x+2}\cdot\frac{x-3}{x-3}$$

Multiply to get common denominators.

$$=\frac{2x+4}{(x+2)(x-3)}-\frac{x-3}{(x+2)(x-3)}$$

Evaluate.

$$=\frac{(2x+4)-(x-3)}{(x+2)(x-3)}$$

Subtract.

$$=\frac{x+7}{(x+2)(x-3)}=\frac{x+7}{x^2-x-6}$$

Simplify the numerator and denominator.

Multiplying Functions

EXAMPLE

Given $f(x)=\dfrac{2}{x-3}$ and $g(x)=\dfrac{1}{x+2}$, express $f(x)g(x)$ as a rational function.

SOLUTION

$$f(x)g(x) =$$

$$= \frac{2}{x-3}\cdot\frac{1}{x+2}$$

Replace $f(x)$ and $g(x)$ with the given functions.

$$= \frac{2\cdot 1}{(x-3)(x+2)}$$

Multiply the numerators and denominators.

$$= \frac{2}{x^2-x-6}$$

Evaluate.

Dividing Functions

EXAMPLE

Given $f(x)=\dfrac{2}{x-3}$ and $g(x)=\dfrac{1}{x+2}$, express $\dfrac{f(x)}{g(x)}$ as a rational function.

SOLUTION

$$\frac{f(x)}{g(x)}=$$

$$=\frac{\dfrac{2}{x-3}}{\dfrac{1}{x+2}}$$

Replace $f(x)$ and $g(x)$ with the given functions.

$$=\frac{2}{x-3}\cdot\frac{x+2}{1}$$

Rewrite as a product (multiply by reciprocal of denominator).

$$=\frac{2(x+2)}{(x-3)1}$$

Multiply the numerators and denominators.

$$=\frac{2x+4}{x-3}$$

Evaluate.

Composition of Functions

EXAMPLE

Table 1 shows a conversion table for men's hat sizes for three countries. The function $g(x) = 8x + 1$ converts from British sizes to French sizes, and the function $f(x) = \frac{1}{8}x$ converts from French sizes to U.S. sizes. Determine the function $h(x) = f(g(x))$ and give its interpretation.

TABLE 1 Conversion Table for Men's Hat Sizes

Britain	$6\frac{1}{2}$	$6\frac{5}{8}$	$6\frac{3}{4}$	$6\frac{7}{8}$	7	$7\frac{1}{8}$	$7\frac{1}{4}$	$7\frac{3}{8}$
France	53	54	55	56	57	58	59	60
United States	$6\frac{5}{8}$	$6\frac{3}{4}$	$6\frac{7}{8}$	7	$7\frac{1}{8}$	$7\frac{1}{4}$	$7\frac{3}{8}$	$7\frac{1}{2}$

SOLUTION

$h(x) = f(g(x))$ This is what we will determine.

$= \frac{1}{8} \cdot g(x)$ In the function f, replace each occurrence of x with $g(x)$.

$= \frac{1}{8} \cdot (8x + 1)$ Replace $g(x)$ with $8x + 1$.

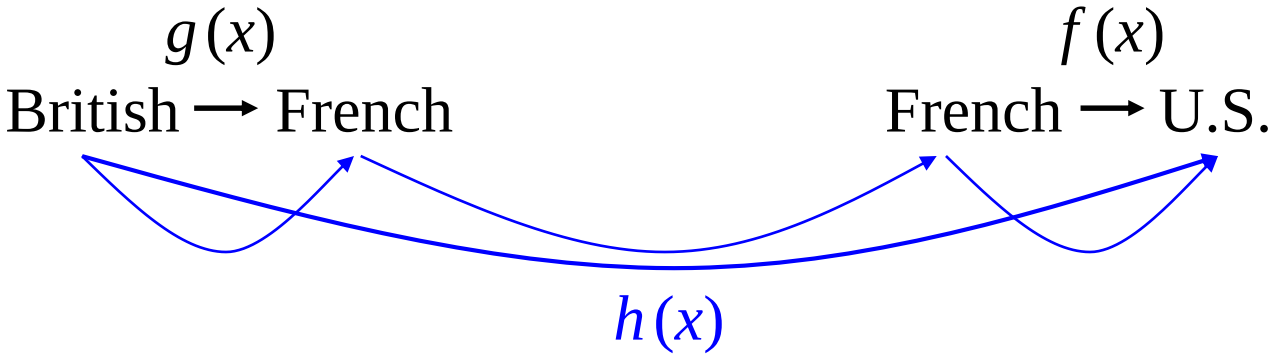
Composition of Functions

CONTINUED

$$= \frac{1}{8} \cdot 8x + \frac{1}{8} \cdot 1 \qquad \text{Distribute.}$$

$$= x + \frac{1}{8} \qquad \text{Multiply.}$$

Therefore, $h(x) = f(g(x)) = x + 1/8$. Now to determine what this function $h(x)$ means, we must recognize that if we plug a number into the function, we may first evaluate that number plugged into the function $g(x)$. Upon evaluating this, we move on and evaluate *that* result in the function $f(x)$. This is illustrated as follows.



Therefore, the function $h(x)$ converts a men's British hat size to a men's U.S. hat size.

Composition of Functions

EXAMPLE

Given $f(x)=\dfrac{2}{x-3}$ and $g(x)=\dfrac{1}{x+2}$.

Find $(f \circ g)(x) = f(g(x))$ and $(g \circ f)(x) = g(f(x))$. Simplify your answer.

SOLUTION

$(f \circ g)(x) = f(g(x)) =$

$$=\frac{2}{g(x)-3}$$

Replace x by $g(x)$ in the function $f(x)$

$$=\frac{2}{\dfrac{1}{x+2}-3}$$

Substitute.

$$=\frac{2}{\dfrac{1}{x+2}-3}\cdot\frac{x+2}{x+2}$$

Multiply the numerators and denominators by $x + 2$.

$$=\frac{2x+4}{-3x-5}$$

Simplify.

Composition of Functions

CONTINUED

$$(g \circ f)(x) = g(f(x)) =$$

$$= \frac{1}{f(x)+2}$$

Replace x by $f(x)$ in the function $g(x)$

$$= \frac{1}{\frac{2}{x-3}+2}$$

Substitute.

$$= \frac{1}{\frac{2}{x-3}+2} \cdot \frac{x-3}{x-3}$$

Multiply the numerators and denominators by $x + 2$.

$$= \frac{x-3}{2x-4}$$

Simplify.

Zeros of Functions

Definition	Example
Zero of a Function: For a function $f(x)$, all values of x such that $f(x) = 0$.	$f(x) = x^2 - 1$ $0 = x^2 - 1$ $x = \pm 1$

Zeros of Functions

Definition	Example
Zero of a Function: Using Factorization, $(x-a)(x-b) = x^2-(a+b)x+ab$	$x^2-5x+6=0$ $(x-2)(x-3)=0$ $x=2,\ x=3$

Zeros of Functions

Definition	Example
Quadratic Formula: A formula for solving any quadratic equation of the form $$ax^2 + bx + c = 0$$ The solution is: $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ There is no solution if $$b^2 - 4ac < 0$$	$$x^2 + 3x - 2 = 0$$ $$a = 1; \quad b = 3; \quad c = -2$$ $$x = \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(-2)}}{2(1)}$$ $$x = \frac{-3 \pm \sqrt{17}}{2}$$ These are the solutions/zeros of the quadratic function $$f(x) = x^2 + 3x - 2$$

Graphs of Intersecting Functions

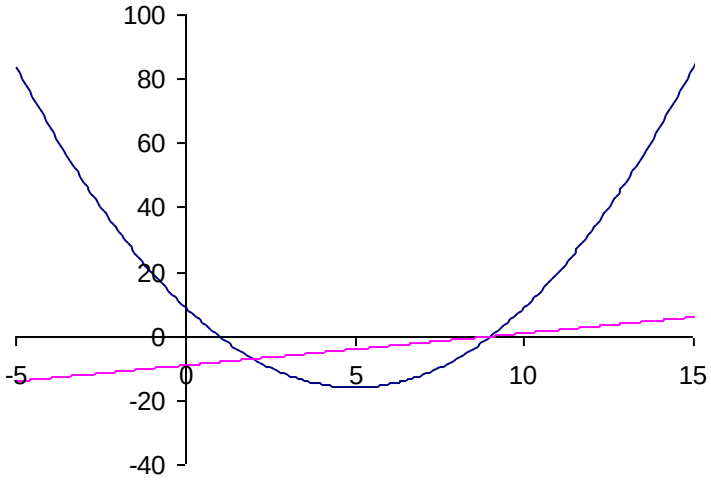
EXAMPLE

Find the points of intersection of the pair of curves.

$$y = x^2 - 10x + 9; \quad y = x - 9$$

SOLUTION

The graphs of the two equations can be seen to intersect in the following graph. We can use this graph to help us to know whether our final answer is correct.



Graphs of Intersecting Functions

CONTINUED

To determine the intersection points, set the equations equal to each other, since they both equal the same thing: y.

$$x^2 - 10x + 9 = x - 9$$

Now we solve the equation for x using the quadratic formula.

$x^2 - 10x + 9 = x - 9$	This is the equation to solve.
$x^2 - 11x + 9 = -9$	Subtract x from both sides.
$x^2 - 11x + 18 = 0$	Add 9 to both sides.
$(x-2)(x-9) = 0$	$x = 2$ or $x = 9$

OR, $a = 1$, $b = -11$, and $c = 18$.

$$x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4(1)(18)}}{2(1)}$$ Use the quadratic formula.

$$x = \frac{11 \pm \sqrt{121 - 72}}{2}$$ Simplify.

Graphs of Intersecting Functions

CONTINUED

$$x = \frac{11 \pm \sqrt{49}}{2}$$ Simplify.

$$x = \frac{11 \pm 7}{2}$$ Simplify.

$$x = \frac{11+7}{2}, \ \frac{11-7}{2}$$ Rewrite.

$$x = 9, \ 2$$ Simplify.

We now find the corresponding y-coordinates for $x = 9$ and $x = 2$. We can use *either* of the original equations. Let's use $y = x - 9$.

$\underline{x = 9}$

$y = x - 9$

$y = 9 - 9$

$y = 0$

$\underline{x = 2}$

$y = x - 9$

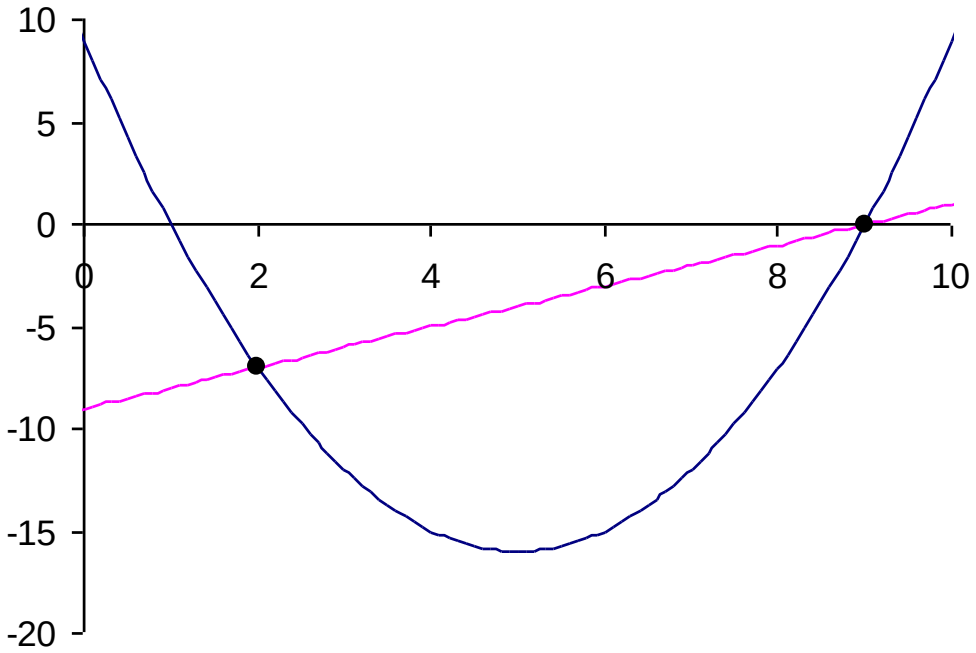
$y = 2 - 9$

$y = -7$

Graphs of Intersecting Functions

CONTINUED

Therefore the solutions are (9, 0) and (2, −7). This seems consistent with the two intersection points on the graph. A zoomed in version of the graph follows.



Factoring

EXAMPLE

Solve the cubic equation $6x - 2x^3 = 0.$

SOLUTION

$6x - 2x^3$ This is the given polynomial.

$2x(3 - x^2)$ Factor $2x$ out of each term.

$2x\left(\sqrt{3}^2 - x^2\right)$ Rewrite 3 as $\sqrt{3}^2$.

Now I can use the factorization pattern: $a^2 - b^2 = (a - b)(a + b)$.

$2x\left(\sqrt{3} - x\right)\left(\sqrt{3} + x\right)$ Rewrite $\sqrt{3}^2 - x^2$ as $\left(\sqrt{3} - x\right)\left(\sqrt{3} + x\right)$.

$2x\left(\sqrt{3} - x\right)\left(\sqrt{3} + x\right) = 0$ Solve the equation.

$x = 0,\ x = \pm\sqrt{3}$

Factoring

EXAMPLE

Solve the equation for x:

$$1 = \frac{5}{x} + \frac{6}{x^2}$$

SOLUTION

$1 = \frac{5}{x} + \frac{6}{x^2}$	This is the given equation.
$x^2(1) = \left(\frac{5}{x} + \frac{6}{x^2}\right)x^2$	Multiply everything by the LCD: x^2.
$x^2 = \frac{5}{x} \cdot x^2 + \frac{6}{x^2} \cdot x^2$	Distribute.
$x^2 = 5x + 6$	Multiply.
$x^2 - 5x - 6 = 0$	Subtract $5x + 6$ from both sides.
$(x+1)(x-6) = 0$	Factor.
$x+1=0 \qquad x-6=0$	Set each factor equal to zero.
$x=-1 \qquad x=6$	Solve.

Exponents

Definition	Example
$b^n = \underbrace{b \cdot b \cdots b}_{n \text{ times}}$	$5^3 = 5 \cdot 5 \cdot 5$
$b^{\frac{1}{n}} = \sqrt[n]{b}$	$5^{\frac{1}{3}} = \sqrt[3]{5}$

Exponents

Definition	Example
$b^{\frac{m}{n}} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$	$5^{\frac{3}{4}} = \sqrt[4]{5^3} = \left(\sqrt[4]{5}\right)^3$
$b^{-\frac{m}{n}} = \frac{1}{b^{\frac{m}{n}}} = \frac{1}{\sqrt[n]{b^m}} = \frac{1}{\left(\sqrt[n]{b}\right)^m}$	$5^{-\frac{3}{4}} = \frac{1}{5^{\frac{3}{4}}} = \frac{1}{\sqrt[4]{5^3}} = \frac{1}{\left(\sqrt[4]{5}\right)^3}$

Exponents

Definition	Example
$b^r \cdot b^s = b^{r+s}$	$6^{\frac{1}{3}} \cdot 6^{\frac{2}{3}} = 6^{\frac{1}{3}+\frac{2}{3}} = 6^{\frac{3}{3}} = 6^1 = 6$
$b^{-r} = \frac{1}{b^r}$	$4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$

Exponents

Definition	Example
$\frac{b^r}{b^s} = b^{r-s}$	$\frac{7^{\frac{4}{3}}}{7^{\frac{1}{3}}} = 7^{\frac{4}{3}-\frac{1}{3}} = 7^{\frac{3}{3}} = 7^1 = 7$
$\left(b^r\right)^s = b^{rs}$	$\left(9^{\frac{4}{5}}\right)^{\frac{5}{8}} = 9^{\frac{4}{5}\cdot\frac{5}{8}} = 9^{\frac{4}{8}} = 9^{\frac{1}{2}} = \sqrt{9} = 3$

Exponents

Definition	Example
$(ab)^r = a^r \cdot b^r$	$(125 \cdot 27)^{1/3} = 125^{1/3} \cdot 27^{1/3}$ $= \sqrt[3]{125} \cdot \sqrt[3]{27} = 5 \cdot 3 = 15$
$\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$	$\frac{10^4}{5^4} = \left(\frac{10}{5}\right)^4 = (2)^4 = 16$

Applications of Exponents

EXAMPLE

Use the laws of exponents to simplify the algebraic expression

$$\frac{\left(-27x^5\right)^{2/3}}{\sqrt[3]{x}}$$

SOLUTION

$$\frac{\left(-27x^5\right)^{2/3}}{\sqrt[3]{x}}$$

This is the given expression.

$$\frac{\left(-27\right)^{2/3}\cdot\left(x^5\right)^{2/3}}{\sqrt[3]{x}}$$

$$\left(ab\right)^r = a^r\cdot b^r$$

$$\frac{\left(-27\right)^{2/3}\cdot\left(x^5\right)^{2/3}}{x^{1/3}}$$

$$b^{\frac{1}{n}} = \sqrt[n]{b}$$

$$\frac{\left(-27\right)^{2/3}\cdot x^{10/3}}{x^{1/3}}$$

$$\left(b^r\right)^s = b^{rs}$$

$$\frac{\left(\sqrt[3]{-27}\right)^2\cdot x^{10/3}}{x^{1/3}}$$

$$b^{\frac{m}{n}} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$$

Applications of Exponents

CONTINUED

$$\frac{(-3)^2 \cdot x^{10/3}}{x^{1/3}}$$

$$\frac{9x^{10/3}}{x^{1/3}}$$

$$9x^{10/3-1/3}$$

$$9x^{9/3}$$

$$9x^3$$

$$\sqrt[3]{-27} = -3$$

$$(-3)^2 = 9$$

$$\frac{b^r}{b^s} = b^{r-s}$$

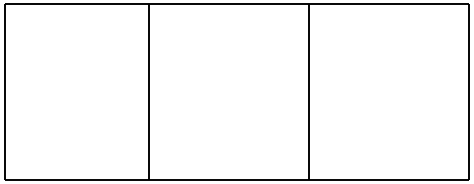
Subtract.

Divide.

Geometric Problems

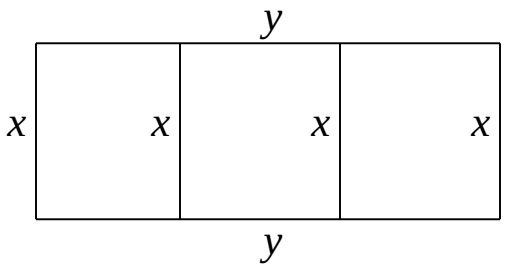
EXAMPLE

Consider a rectangular corral with two partitions, as shown below. Assign letters to the outside dimensions of the corral. Write an equation expressing the fact that the corral has a total area of 2500 square feet. Write an expression for the amount of fencing needed to construct the corral (including both partitions).



SOLUTION

First we will assign letters to represent the dimensions of the corral.



Geometric Problems

CONTINUED

Now we write an equation expressing the fact that the corral has a total area of 2500 square feet. Since the corral is a rectangle with outside dimensions x and y, the area of the corral is represented by:

$$A = xy$$

Now we write an expression for the amount of fencing needed to construct the corral (including both partitions). To determine how much fencing will be needed, we add together the lengths of all the sides of the corral (including the partitions). This is represented by:

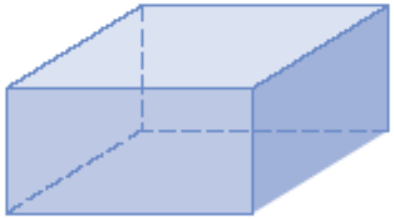
$$F = x + x + x + x + y + y$$

$$F = 4x + 2y$$

Surface Area

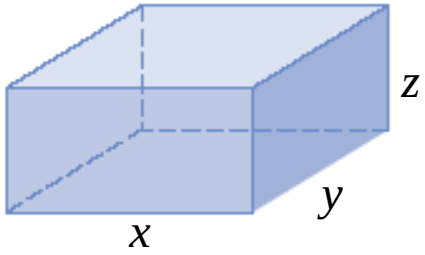
EXAMPLE

Assign letters to the dimensions of the geometric box and then determine an expression representing the volume and the surface area of the open top box.



SOLUTION

First we assign letters to represent the dimensions of the box.

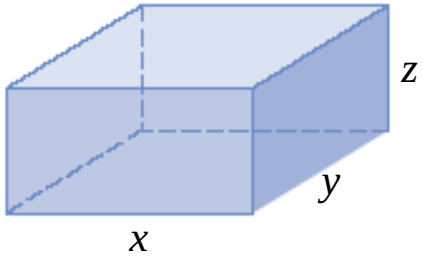


Therefore, an expression that represents the volume is:

$$V = xyz.$$

Surface Area

CONTINUED



Now we determine an expression for the surface area of the box. Note, the box has 5 sides which we will call **Left** (L), **Right** (R), **Front** (F), **Back** (B), and **Bottom** (Bo). We will find the area of each side, *one at a time*, and then add them all up.

L: yz R: yz

F: xz B: xz

Bo: xy

Therefore, an expression that represents the surface area of the box is:

$$S = yz + yz + xz + xz + xy = 2yz + 2xz + xy.$$